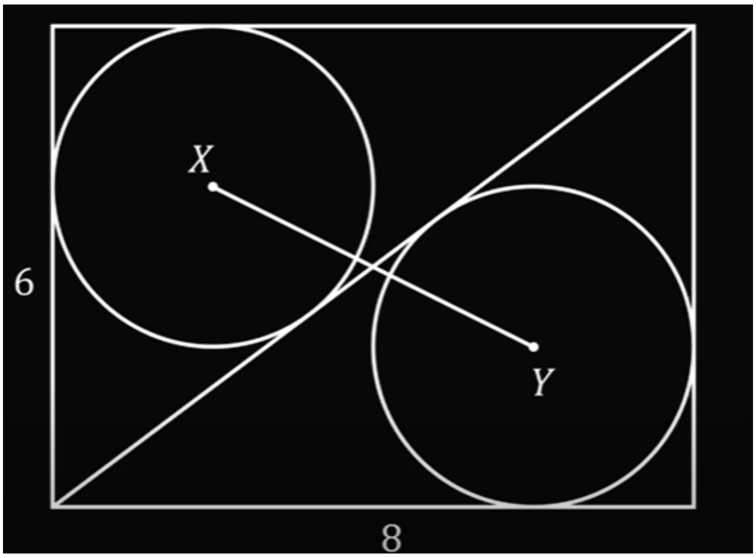
October Math Gems

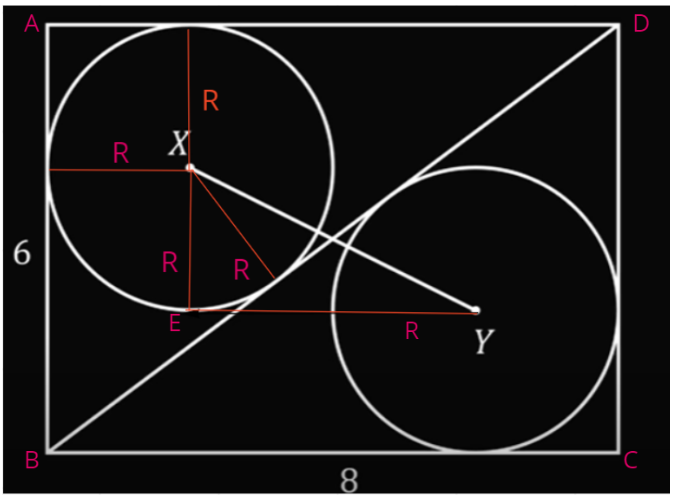
Problem of the week 19

§1 problems

Problem 1.1. What is the length of $|XY|$



answer. First, we can see that the length of $\overline{EY}$ is $8-2R$ and the length of $\overline{XE}$ is $6-2R$. Therefore the length of $\overline{XY}$ is equal to $\sqrt{(8-2R)^2+(6-2R)^2}$. Second, we can see that the area of triangle BCD is equal to 24, and the area of triangle ABC is equal to the areas of the triangles AXB, AXD, and XDB. Their areas are $3R$, $4R$, and $5R$. Therefore, $3R+4R+5R=24$. So, R is 2. We can substitute now in the first equation and get that the length of $\overline{XY}$ is $\sqrt{20}$.



$\square$

Problem 1.2. Suppose that α, β, and γ are the roots of the equation $x^3+3x^2-24x+1=0$. Find the value $\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\gamma}$

answer. First, we need to try to factorize the equation. So, we can add $27x$ to both sides, and factorize it, which gives us the equation $(x+1)^3 = 27x$. Taking the cubic root to both sides we get that $\frac{x+1}{3} = \sqrt[3]{x}$. Since α, β, and γ are roots of the original equation, then they satisfy the equation we just got. Therefore, $\frac{\alpha+1}{3} = \sqrt[3]{\alpha}$, $\frac{\beta+1}{3} = \sqrt[3]{\beta}$, and $\frac{\gamma+1}{3} = \sqrt[3]{\gamma}$. Summing these all up, we get $\frac{\alpha+\beta+\gamma+3}{3} = \sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\gamma}$. Using the fact that the sum of the roots of the cubic equation is $\frac{-b}{a}$, we see that $\alpha+\beta+\gamma = \frac{-3}{1} = -3$. Therefore, $\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\gamma} = \frac{-3+3}{3} = 0$ $\square$

Problem 1.3. In the equation ax^2+5x+2, solve for a so that the equation has exactly one solution

answer. First, we know that a quadratic function has only on solution if its determinant is equal to 0. Therefore, $b^2-4ac = 0$ and a is equal to $\frac{25}{8}$. However, this is not the only solution because if $a = 0$, the the function becomes linear and has only one solution. So, the values for a are 0 and $\frac{25}{8}$ $\square$

Problem 1.4. Given that $\frac{1}{\sqrt[3]{25}+\sqrt[3]{5}+1} = A\sqrt[3]{25}+B\sqrt[3]{5}+C$. So, what is the value of $A+B+C$ (A,B, and C are rational numbers).

answer. Let x be equal to $\sqrt[3]{5}$. We can, then, write the fraction as $\frac{1}{x^2+x+1}$. We, then, multiply by $\frac{x-1}{x-1}$. We arrive to the following fraction $\frac{x-1}{(x^2+x+1)(x-1)}$, and we can factorize the denominator to be $\frac{x-1}{(x-1)^3}$. This is equal to $\frac{\sqrt[3]{5}-1}{4} = \frac{1}{4}\sqrt[3]{5}-\frac{1}{4}$. Therefore, $A = 0$, $B = \frac{1}{4}$, and $C = \frac{-1}{4}$ $\square$

Problem 1.5. The leg of a right-angled triangle is equal to $\frac{1}{5}$ the sum of the other sides, and the triangle's perimeter is 1. What is the area of the triangle?

answer. Let us say the sides of the triangle are a, b, and c. Therefore, $a+b+c = 1$ and $b = \frac{1}{5}(a+c)$. Substituting this value of b in the first equation, we get the value of b to be $\frac{1}{6}$. Using the Pythagorean theorem, we can substitute with the value of b and substitute with $c^2 = (\frac{5}{6}-a)^2$. We get that $a = \frac{2}{5}$. Therefore, the area of the triangle is $\frac{1}{30}$ $\square$

Problem 1.6. There are 90 equally spaced dots marked on a circle. Shannon chooses an integer, n. Beginning at a randomly chosen dot, Shannon goes around the circle clockwise and colours in every nth dot. He continues going around and around the circle colouring in every nth dot, counting each dot whether it is coloured in or not, until he has coloured in every dot. Which of the following could have been Shannon's integer ? $(3, 5, 6, 7)$

answer. First, we can eliminate any even number since Shanon won't color any odd number in this way. Second, we can also eliminate 3 and 5 since they are divisible by 90, which means he will come to 90 and go over the same path again. Therefore, the only possible choice is 7 $\square$

Problem 1.7. If $x^2 = 2023+y$, $y^2 = 2023+x$, where $x \neq y$ and x and y are both real numbers. Find the value of xy

answer. First, it is clear that

$$x^2 - y = y^2 - x$$

We can rewrite this as

$$(x-y)(x+y+1) = 0$$

This gives the possibilities of either $x = y$ or $x + y = -1$; however, $x = y$ contradicts the givens, which eliminates it. We can substitute x and y with the values

$$x = -1 - y$$

and

$$y = -1 - x$$

We get the following equations

$$x^2 + x - 2022 = 0$$

$$y^2 + y - 2022 = 0$$

We can say that x and y are roots of some function

$$p^2 + p - 2022 = 0$$

Therefore, their product is equal to

$$\frac{-2022}{1} = -2022$$

□

Problem 1.8. Solve for real values of x

$$2\sqrt[3]{2x+1} = x^3 - 1$$

answer. First, let $\sqrt[3]{2x+1}$ be equal to y. Therefore, we can write y in terms of x to be $y = \frac{x^3-1}{2}$, and we can write x in terms of y to be $\frac{y^3-1}{2}$. These imply that $x = y$, which can be proved geometrically or by contradiction (Hint: Try to see the cases where $x > y$ and $y > x$). Therefore, we can write the cubic function

$$x^3 - 2x - 1 = 0$$

This cubic function has the roots of -1 and $\frac{1 \pm \sqrt{5}}{2}$, which is our answer □

Problem 1.9. Simplify and prove your answer for real values of x

$$x = \sqrt[3]{8 + 3\sqrt{21}} + \sqrt[3]{8 - 3\sqrt{21}}$$

answer. First, let $8 + 3\sqrt{21} = a$ and $8 - 3\sqrt{21} = b$. Therefore, we can cube both sides of the equation and get

$$x^3 = a + b + 3\sqrt[3]{a^2 b} + 3\sqrt[3]{b^2 a}$$

We can write the value ab as $(-5)^3$ (since the value ab is the difference between two squares) After substituting and simplifying, we get

$$x^3 = a + b - 15(\sqrt[3]{a}\sqrt[3]{b})$$

Since $x = \sqrt[3]{a}\sqrt[3]{b}$, we can rewrite the equation as a polynomial of the third degree

$$x^3 + 15x - 16 = 0$$

This polynomial has only one real solution which is 1. So, $x = 1$ □

Problem 1.10. If in triangle ABC we have

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0,$$

show that $\sin^3 A + \sin^3 B + \sin^3 C = 3 \sin A \sin B \sin C$.

answer. Since $\det \left(\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \right) = 0$, we have $a(bc - a^2) + b(ca - b^2) + c(ab - c^2) = 0 \implies a^3 + b^3 + c^3 - 3abc = 0$. From the Law of Sines,

$$k = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Substituting $a = k \sin A, b = k \sin B, c = k \sin C$ gives us

$$k^3(\sin^3 A + \sin^3 B + \sin^3 C) = 3k^3 \sin A \sin B \sin C$$

Therefore,

$$\sin^3 A + \sin^3 B + \sin^3 C = 3 \sin A \sin B \sin C$$

as desired. □

Problem 1.11. if ABC is a triangle and $\frac{AB}{cosC} = \frac{BC}{cosA} = \frac{AC}{cosB}$ then prove that ABC is equilateral.

answer. Looks suspiciously like Law of Sines, but with cosines instead. It would be prudent to use Law of Sines then, perhaps the fact that $\frac{a}{b} = \frac{\sin A}{\sin B}$? But from the problem, we also have that $\frac{a}{b} = \frac{\cos A}{\cos B}$. we have $\dfrac{\sin A}{\sin B} = \dfrac{\cos A}{\cos B}$ $\cos B \sin A - \cos A \sin B = 0$ $\sin(A - B) = 0$ Therefore, $A = B$, and by symmetry, $A = B = C$. □

Problem 1.12. if $\frac{4\sqrt{x^2+1}}{x} = \frac{5\sqrt{y^2+1}}{y} = \frac{6\sqrt{z^2+1}}{z}$ and $x + y + z = xyz$. find x , y and z.

answer. Per the intital condition, take $x = \tan A$, $y = \tan B$, $z = \tan C$, $A+B+C = \pi$, so that $\frac{4 \sec A}{\tan A} = \frac{5 \sec B}{\tan B} = \frac{6 \sec C}{\tan C} \implies \frac{4}{\sin A} = \frac{5}{\sin B} = \frac{6}{\sin C}$ But this is just a statement of the Law of Sines, so A, B and C are angles opposite the side lengths 4, 5 and 6, respectively, in a triangle. Hence, by the Law of Cosines $\cos A = \frac{6^2+5^2-4^2}{2(6)(5)} = \frac{3}{4} \implies \sin A = \frac{\sqrt{7}}{4}$ $\cos B = \frac{6^2+4^2-5^2}{2(6)(4)} = \frac{9}{16} \implies \sin A = \frac{5\sqrt{7}}{16}$ $\cos C = \frac{5^2+4^2-6^2}{2(6)(5)} = \frac{1}{8} \implies \sin C = \frac{3\sqrt{7}}{8}$ Therefore, $x = \tan A = \frac{\sqrt{7}}{3}$, $y = \tan B = \frac{5\sqrt{7}}{9}$, $z = \tan C = 3\sqrt{7}$. Finally, $\boxed{(x, y, z) = \left(\dfrac{\sqrt{7}}{3}, \dfrac{5\sqrt{7}}{9}, 3\sqrt{7} \right)}$. □

Problem 1.13. Solve in real numbers the following equation: $(\sqrt{6} - \sqrt{5})^x + (\sqrt{3} - \sqrt{2})^x + (\sqrt{3} + \sqrt{2})^x + (\sqrt{6} + \sqrt{5})^x = 32$.

answer. Let $a = \sqrt{6} - \sqrt{5}$ and $b = \sqrt{3} - \sqrt{2}$ and $f(x) = a^x + a^{-x} + b^x + b^{-x} - 32$ $f(x)$ is an even function increasing over $\mathbb{R}^+$ and so decreasing over $\mathbb{R}^-$ So at most one real non negative root and its opposite. And since $f(2) = 0$ we get the answer $\boxed{x \in \{-2, +2\}}$ □

Problem 1.14. Solve in $\mathbb{R}$ the equation

$$\frac{1}{2^x+x+1}+\frac{1}{3^x-4x-3}=\frac{1}{2^x-4x-2}+\frac{1}{x+3^x}$$

.

answer. Common denominator on both sides yields

$$\frac{3^x-3x-2+2^x}{(2^x+x+1)(3^x-4x-3)}=\frac{3^x-3x-2+2^x}{(2^x-4x-2)(x+3^x)}$$

We see that the numerators are equal, meaning that this equality holds when the numerator is equivalent to 0, or when the denominators are equal and not 0. Checking the first case, $3^x-3x-2+2^x=0$ in real numbers when $x=0,1$. The second case, we solve as

$$(2^x+x+1)(3^x-4x-3)=(2^x-4x-2)(x+3^x)$$

$$(2^x)(3^x)-(2^x)(4x)-(3)(2^x)+(x)(3^x)-4x^2-3x+3^x-4x-3=(2^x)(3^x)-(4x)(3^x)-(2)(3^x)+(x)(2^x)-4x^2-2x$$

$$(4x)(3^x)-(4x)(2^x)+(x)(3^x)-(x)(2^x)+(3)(3^x)-(3)(2^x)-5x-3=0$$

$$(3^x-2^x-1)(5x+3)=0$$

This is true when $x=-\frac{3}{5},1$. Checking our answers through substitution verifies that our solutions are indeed $\boxed{x=0,1,-\frac{3}{5}}$. □

Problem 1.15. Find all real numbers a,b,c,d such that

$$\begin{cases} a+b+c+d=20, \\ ab+ac+ad+bc+bd+cd=150. \end{cases}$$

answer. Let $S=ab+ac+ad+bc+bd+cd=150$. We know that $(a+b+c+d)^2=a^2+b^2+c^2+d^2+2S$, so $a^2+b^2+c^2+d^2=400-300=100$.
So we have to find all real numbers a,b,c,d such that (1) $a+b+c+d=20$, and (2) $a^2+b^2+c^2+d^2=100$.
By AM-QM inequality we have that $\frac{a+b+c+d}{4}\le\sqrt{\frac{a^2+b^2+c^2+d^2}{4}}$, with equality if and only if $a=b=c=d$.
But these expressions are in fact equal, as they both equal 5. So the only solution is $a=b=c=d=5$ □

Problem 1.16. Let ABC be a right-angled triangle with $\angle ABC=90°$, and let D be on AB such that $AD=2DB$. What is the maximum possible value of $\angle ACD$?

answer. let $BD=1, AD=2$. Consider a ray with vertex B perpendicular to AB. Let O be a point on the perpendicular bisector of AD, starting at its midpoint and moving in the direction of the ray. Considering the circle centered at O through A,D, all points on major arc AD of that circle result in the same value of $\angle ACD$, and this angle decreases as O moves further away from AD, so our maximum value occurs when the circle is first tangent to our ray. In this case, by PoP $BC^2=BD\cdot BA=3\implies BC=\sqrt{3}\implies \angle ACD=\boxed{30°}$ □

Problem 1.17. In a triangle ABC, $\angle A=2\cdot\angle B$. Prove that $a^2=b(b+c)$.

answer. Produce CA to K such that $CK = (b+c)$.So, $AK = c = AB$. From that it follows that $\angle AKB = \angle ABK = \angle B$.So, we have $\triangle CBK \sim \triangle CAB \implies \frac{CK}{CB} = \frac{CB}{CA} \implies \frac{(b+c)}{b} = \frac{a}{b} \implies b(b+c) = a^2$ □

Problem 1.18.

Prove that :

$$\sin^2\alpha - \sin^2\beta = \sin(\alpha+\beta)\sin(\alpha-\beta)$$

answer. Using the product formulas,

$$L.H.S = \sin(\alpha+\beta)\sin(\alpha-\beta) = \frac{1}{2}(\cos(2\beta) - \cos 2\alpha)$$

Then, using the double angle formulas, $\frac{1}{2}(\cos(2\beta) - \cos 2\alpha) = \frac{1}{2}(\cos^2\beta - \sin^2\beta - \cos^2\alpha + \sin^2\alpha)$
Thus, the required identity is equivalent to proving that,

$$2\sin^2\alpha - 2\sin^2\beta = \cos^2\beta - \sin^2\beta - \cos^2\alpha + \sin^2\alpha$$

But, notice that this is equivalent to proving

$$\sin^2\alpha - \sin^2\beta = \cos^2\beta - \cos^2\alpha \implies \sin^2\alpha + \cos^2\alpha = \sin^2\beta + \cos^2\beta$$

which is clearly true as from the Pythagorean Identities it is equivalent to,

$$1 = 1$$

which is trivially true. □

Problem 1.19. Determine all pairs (a,b) of real numbers for which there exists a unique symmetric 2×2 matrix M with real entries satisfying trace$(M) = a$ and $\det(M) = b$.

answer. We look at the matrix $M = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$.
We want $x + z = a$ and $xz - y^2 = b$
So x, z are roots of $u^2 - au + (y^2+b) = 0$
Thus $x = \frac{a\pm\sqrt{a^2-4(y^2+b)}}{2}$ and $y = \frac{a\mp\sqrt{a^2-4(y^2+b)}}{2}$
Solutions can only be unique if $a^2 = 4(y^2+b) \implies y = \pm\frac{\sqrt{a^2-4b}}{2}$ where we must have $x = z = \frac{a}{2}$.
However, y will only be unique if $a^2 = 4b$, so the solution set is $(a,b) = (2t, t^2)$ where $t \in \mathbb{R}$. □

Problem 1.20. Determine all pairs (a,b) of non-negative integers such that

$$\frac{a+b}{2} - \sqrt{ab} = 1.$$

answer. let $a \geq b$. Let $x = \sqrt{a}$ and $y = \sqrt{b}$. Then

$$\frac{x^2+y^2}{2} - xy = 1$$
$$\frac{(x-y)^2}{2} = 1$$
$$(x-y)^2 = 2$$
$$x - y = \sqrt{2}$$

. Then

$$\sqrt{a} = \sqrt{2} + \sqrt{b}$$
$$a = b + 2\sqrt{2b} + 2$$

. But $a, b, 2$ are integers, so $2\sqrt{2b}$ is an integer, which means $b = 2k^2$ for some non-negative integer k, and

$$\begin{aligned}\sqrt{a} &= \sqrt{2} + \sqrt{b} \\ &= (k+1)\sqrt{2}\end{aligned}$$

So $a = 2(k+1)^2$. Notice that

$$\frac{a+b}{2} - \sqrt{ab} = \frac{2(k+1)^2 + 2k^2}{2} - \sqrt{2(k+1)^2 \cdot 2k^2} = 2k^2 + 2k + 1 - 2(k+1)k = 1,$$

so it indeed works. Therefore, all such pairs are those of the form $(2(k+1)^2, 2k^2)$ or $(2k^2, 2(k+1)^2)$, for any non-negative integer k. $\square$